AIAA JOURNAL
Vol. 32, No. 11, November 1994

Logarithmic Laws for Compressible Turbulent Boundary Layers

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Dimensional similarity arguments proposed by Millikan are used with the Morkovin hypothesis to deduce logarithmic laws for compressible turbulent boundary layers as an alternative to the traditional van Driest analysis. It is shown that an overlap exists between the wall layer and the defect layer, and this leads to logarithmic behavior in the overlap region. The von Karman constant is found to depend parametrically on the Mach number based on the friction velocity, the dimensionless total heat flux, and the specific heat ratio. Even though it remains constant at approximately 0.41 for a freestream Mach number range of 0–4.544 with adiabatic wall boundary conditions, it rises sharply as the Mach number increases significantly beyond 4.544. The intercept of the logarithmic law of the wall is found to depend on the Mach number based on the friction velocity, the dimensionless total heat flux, the Prandtl number evaluated at the wall, and the specific heat ratio. On the other hand, the intercept of the logarithmic defect law is parametric in the pressure gradient parameter and all of the aforementioned dimensionless variables except the Prandtl number. A skin friction law is also deduced for compressible boundary layers. The skin friction coefficient is shown to depend on the momentum thickness Reynolds number, the wall temperature ratio, and all of the other parameters already mentioned.

Introduction

SINCE the original proposal of the van Driest I density-weighted transformation[1] for the mean velocity, and its use to justify the extension of the incompressible law of the wall to compressible boundary layers, a lively discussion has ensued regarding the applicability of the law of the wall to high Mach number flows with thermal wall boundary conditions other than adiabatic. Recent calculations with two-equation models[2] and second-order closure models,[3] when compared with measurements,[4–6] seem to suggest that the van Driest compressible law of the wall may only be applicable to flows with fairly low Mach numbers. Furthermore, the von Karman constant κ was found in these studies to be 0.41 when the untransformed mean velocity is plotted in wall coordinates; it was a constant only in the case of compressible flows over an adiabatic wall with external Mach numbers sufficiently less than 10. On the other hand, Huang et al.[7] and Huang and Coleman[8] have argued that the van Driest I transformation is valid for compressible flows with a fairly wide range of different Mach numbers and thermal wall boundary conditions. They further proposed that κ in the van Driest compressible law of the wall is equal to 0.41, and the intercept also remains fairly constant at 5.2 for the range of Mach numbers studied. The starting point of their analysis is inner-layer similarity, thus leading to mixing length formulas for the mean velocity and mean temperature. A further assumption of a constant turbulent Prandtl number allows the mean temperature to be expressed in terms of the mean velocity. The logarithmic behavior of the mean velocity and temperature, therefore, follows directly from the mixing length formula invoked for the velocity field. In their analysis, a constant κ is tacitly assumed and is built into the mixing length formula used to derive the subsequent logarithmic law of the wall. The generalization to cover the whole layer was carried out by applying Coles' law of the wake for the outer region with a van Driest type damping of the mixing length for the viscous layer. This approach tacitly assumes the validity of the Markovin hypothesis,[9] and thus allows the wake function derived from incompressible boundary-layer data to be directly extended to compressible flows. They, however, did not present evidence to show that an overlap exists between the wall layer and the defect layer with the transformed mean velocity. According to Millikan,[10] the presence of an overlap is a necessary condition for the logarithmic behavior to exist in any wall-bounded turbulent flow.

The objective of this paper is to invoke the Morkovin hypothesis[9] and the dimensional similarity arguments of Millikan[10] to demonstrate that the assumption of an overlap between the wall layer and the defect layer in flat plate compressible boundary-layer flows, with different wall thermal boundary conditions, leads to logarithmic behavior in the law of the wall and the defect law. Furthermore, this analysis is used to show that κ is a true constant only for isothermal, incompressible flows. In general, for compressible flows, κ is parametric in the Mach number based on the friction velocity, the dimensionless total heat flux, and the specific heat ratio. Comparisons of this result are made with the traditional van Driest compressible law of the wall. Since dimensional arguments cannot be used to determine the exact value of κ, comparisons with measurements[4–6,11–13] are used to establish the approximate values of κ and the intercepts associated with the logarithmic law of the wall and the logarithmic defect law for compressible boundary layers with different wall thermal boundary conditions. Finally, a skin friction law is deduced for compressible boundary layers valid for flows with and without streamwise pressure gradient effects.

Dimensional Similarity Analysis

According to Morkovin,[9] dynamic similarity exists between compressible and incompressible mean turbulent flows. In other words, compressibility effects may be accounted for by the mean density field alone. If this hypothesis is invoked, the dimensional similarity arguments used by Millikan[10] to deduce the incompressible law of the wall can be extended to compressible flows and used to show the existence of a logarithmic region in the mean

Received Oct. 29, 1993; revision received April 29, 1994; accepted for publication May 1, 1994.

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velocity distributions under different wall thermal boundary conditions. For incompressible flows, the mean velocity near a wall is influenced by the wall shear stress τ_w, the local fluid density ρ, the viscosity μ, and the normal distance from the wall y. For compressible flows, using the Morkovin hypothesis, the mean velocity U is again dependent on τ_w, the fluid properties evaluated at the wall (i. e., ρ_w, μ_w, k_w, C_p, and C_v, which are the density, viscosity, thermal conductivity, and specific heats, respectively), the total heat flux Q_{tot}, the wall temperature T_w, and the distance from the wall y. The choice of the fluid properties evaluated at the wall is dictated by the fact that only these fluid properties are constant in the near-wall region and that Q_{tot} is a uniformly valid expression linking the outer layer and the inner layer. Application of the Buckingham Π theorem leads to six dimensionless groupings, and they can be expressed as

$$U^+ = U/u_\tau = F\left(y_w^+, B_q, M_\tau, Pr_w, \gamma\right) \tag{1}$$

where $u_\tau = (\tau_w/\rho_w)^{1/2}$, $y_w^+ = yu_\tau/\nu_w$, $B_q = Q_{tot}/(\rho_w C_p u_\tau T_w)$, $M_\tau = u_\tau/a_w$, $Pr_w = (C_p\mu_w)/k_w$, a_w is the sound speed evaluated at the wall, and $\gamma = C_p / C_v$ is the ratio of the specific heats evaluated at the wall. The choice of M_τ follows from the fact that, for compressible boundary-layer flows, the ratio of the wall shear stress to the static pressure at the wall affects the flow.[14] This ratio can be written as $\tau_w / \gamma p_w$ where p_w is the wall static pressure. Since $u_\tau = (\tau_w/\rho_w)^{1/2}$ and $a_w^2 = \gamma p_w/\rho_w$, it can be easily shown that $\tau_w/\gamma p_w = u_\tau^2/a_w^2 = M_\tau^2$. It should be pointed out that, as long as the temperature is not high enough to give rise to real gas effects such as dissociation, ionization, etc., the quantities γ and Pr_w can be considered constant. However, for the sake of completeness, they are carried along in the following analysis. Using an alternate approach, Rotta[14] obtained a slightly different function F for Eq. (1). In Rotta's function, Pr_w is replaced by $Pr = \mu(C_p/k)$. As a result, another parameter n appears in F, where n is the exponent of the viscosity law assumed, i.e., $\mu = \mu_w(T/T_w)^n$. It is evident from the preceding analysis that the more appropriate Pr is that evaluated at the wall, rather than its local value.

In the defect layer, the drag generated as a result of the presence of the wall acts to slow down the fluid, thus creating a velocity defect. Therefore, for incompressible flow, the velocity defect is independent of the fluid viscosity, but instead depends on the outer variables, such as the boundary-layer thickness and the streamwise pressure gradient. For compressible flows, one could argue that the velocity defect is again independent of the fluid viscosity and thermal conductivity but is still affected by the fluid density. Consequently, the velocity defect in the outer region is found to depend on δ, ρ_w, C_p, C_v, Q_{tot}, T_w, dP/dx, and y. Application of the Π theorem again gives rise to the expression

$$[(U_\infty - U)/u_\tau] = G(\eta, B_q, M_\tau, \gamma, \beta) \tag{2a}$$

or

$$\frac{U}{u_\tau} = \frac{U_\infty}{u_\tau} - G(\eta, B_q, M_\tau, \gamma, \beta) \tag{2b}$$

where U_∞ is the freestream mean velocity, $\beta = (\delta\, dP/dx)/\tau_w$ the pressure gradient parameter [which is related to the Clauser pressure gradient parameter β_c, by $\beta = (\delta/\delta^*)\beta_c$], $\eta = y/\delta$ the dimensionless y coordinate, δ the boundary-layer thickness, δ^* the displacement thickness, P the mean static pressure, and x the streamwise coordinate. Rotta[14] has also proposed a functional form similar to Eq. (2a) for the velocity defect; however, his function only depends on η, M_τ, and a dimensionless parameter related to the location of the virtual maximum temperature. The dependence on γ, β, and B_q is lacking, and it appears that pressure gradient effects are not accounted for explicitly. According to Mellor[15] and Mellor and Gibson,[16] this cannot be the case, even for incompressible boundary layers. Therefore, the present analysis appears to be more physical because it approaches the incompressible limit correctly and leads to a more general expression for the velocity defect.

Millikan[10] pointed out that if an overlap exists between the wall layer and the defect layer, the functions F and G must be logarithmic. The wall logarithmic behavior first deduced by Millikan[10] has been formalized mathematically by Mellor[15] as the inner asymptote of the defect layer, $\eta \to 0$, or the outer asymptote of the wall layer, $y_w^+ \to \infty$. According to Tennekes and Lumley,[17] the dimensionless velocity gradient of Eqs. (1) and (2b) must be equal in the limit as $y_w^+ \to \infty$ and $\eta \to 0$ simultaneously. Therefore, this leads to

$$\frac{y}{u_\tau}\frac{\partial U}{\partial y} = y_w^+ \frac{\partial F}{\partial y_w^+} = -\eta\frac{\partial G}{\partial \eta} = \frac{1}{\kappa(M_\tau, B_q, \gamma)} \tag{3}$$

In the overlap region, both expressions for the velocity gradient must be equal. Even though these expressions are parametric in β, M_τ, Pr_w, B_q, and γ, one is a function of y_w^+ only whereas the other is a function of η alone. These two expressions can be the same if and only if they are equal to a constant κ that is parametric in M_τ, B_q, and γ alone. On integration, the following relations are obtained:

$$U^+ = \frac{1}{\kappa(M_\tau, B_q, \gamma)} \ln y_w^+ + B(B_q, M_\tau, \gamma, Pr_w) \tag{4}$$

$$\frac{U_\infty - U}{u_\tau} = -\frac{1}{\kappa(M_\tau, B_q, \gamma)} \ln \eta + A(\beta, B_q, M_\tau, \gamma) \tag{5}$$

where B is the intercept of the logarithmic law of the wall and A the intercept of the logarithmic defect law. In general, these intercepts are parametric in M_τ, γ, Pr_w, β and B_q. For zero-pressure-gradient compressible flow over an adiabatic wall, β and B_q vanish, and the Prandtl number evaluated at the wall remains constant. If γ is assumed to be fairly constant over the Mach number range considered, then κ, A, and B would be parametric in M_τ only. The dependence of κ, A, and B on M_τ may or may not be strong; however, it has to be determined by comparison with measurements. Therefore, for this simple compressible flow, κ, A, and B are not true constants, and they should be expected to vary with the Mach number. In general, κ is parametric in M_τ and B_q, even if γ is assumed to be constant in the Mach number range considered.

The traditional van Driest compressible law of the wall[1] was formulated using a transformed mean velocity U_c and the incompressible von Kármán constant κ_0. It can be written as

$$\frac{U_c}{u_\tau} = \frac{1}{\kappa_0} \ln y_w^+ + B \tag{6}$$

where

$$U_c = \int_0^U \left(\frac{\rho}{\rho_w}\right)^{1/2} dU\, m \tag{7}$$

and κ_0 is taken to be 0.41. The intercept B was assumed to vary weakly with the Mach number and wall thermal boundary conditions. A straightforward differentiation of Eq. (6), making use of Eq. (7), yields

$$\frac{dU^+}{dy_w^+} = \left(\frac{\rho_w}{\rho}\right)^{1/2} \frac{1}{\kappa_0 y_w^+} = \frac{1}{\kappa_0 (T_w/T)^{1/2} y_w^+} \tag{8}$$

because $\rho RT = \text{const}$ for the zero pressure gradient boundary layer. For adiabatic wall boundary condition, $T_w / T \approx \text{const}$ in the logarithmic region. If this constant is parametric in M_τ and γ (an apparently reasonable assumption), it follows that Eq. (8) is a special case of Eq. (4) wherein $\kappa(M_\tau, \gamma) = \kappa_0(T_w/T_{\log})^{1/2}$. Here, $T_{\log}$ is the temperature evaluated in the log region. Hence, the results derived herein appear to be qualitatively consistent with the van Driest compressible law of the wall for adiabatic boundary conditions.

Mellor and Gibson[16] choose the defect thickness Δ, rather than the boundary-layer thickness δ as the appropriate length scale to use in the defect layer. Their analysis also yields a logarithmic defect law with an intercept given by A_m. For the case of the flat plate boundary layer at zero pressure gradient, Mellor and Gibson[16] determined A_m to be -0.6. Since A_m and A are related by

$$A = A_m + \frac{1}{\kappa} \ln \frac{\Delta}{\delta} \tag{9}$$

Table 1 Comparison of calculated and measured values of C_f, κ, B, and A

									Measurements				k-ε Calculations			
Case	Source	β	T_w/T_{aw}	M_∞	M_τ	Re_θ	$(y_w^+)_{in}$	$(y_w^+)_{out}$	$C_f \times 10^3$	κ	B	A	$C_f \times 10^3$	κ	B	A
a	Klebanoff[12]	0	1.0	0	0	7,800	40	500	2.84	0.43	5.34	2.70	2.87	0.40	3.82	2.48
b	Karlsson and Johansson[11]	0	1.0	0	0	2,420	40	200	3.54	0.41	4.80	2.10	3.53	0.37	3.11	2.26
c	Samuel and Joubert[13]	16.98	1.0	0	0	14,932	30	600	1.35	0.42	5.15	7.00	1.37	0.39	3.53	7.42
d	Samuel and Joubert[13]	33.51	1.0	0	0	24,707	30	500	0.69	0.41	5.70	14.59	0.80	0.42	3.23	15.13
e	Spina and Smits[6]	0	1.11	2.87	0.067	83,889	300	2000	1.10	0.52	6.82	1.02	1.07	0.46	6.01	0.32
f	Fernholz and Finley[4]	0	1.0	2.24	0.063	20,797	66	1000	1.62	0.41	4.04	0.48	1.64	0.42	4.50	0.01
g	Fernholz and Finley[4]	0	1.0	4.54	0.114	5,320	20	200	1.26	0.41	4.00	−0.33	1.30	0.42	4.25	−0.68
h	Fernholz and Finley[4]	0	1.0	10.31	0.113	15,074	40	150	0.24	0.54	6.10	−1.10	0.22	0.47	4.86	−0.68
i	Fernholz and Finley[4]	0	0.92	5.29	0.135	3,936	30	150	1.31	0.48	5.27	−0.49	1.27	0.43	4.62	−1.13
j	Kussoy and Horstman[5]	0	0.30	8.18	0.181	4,600	40	300	0.98	0.35	6.48	−1.38	0.91	0.25	0.54	−2.28

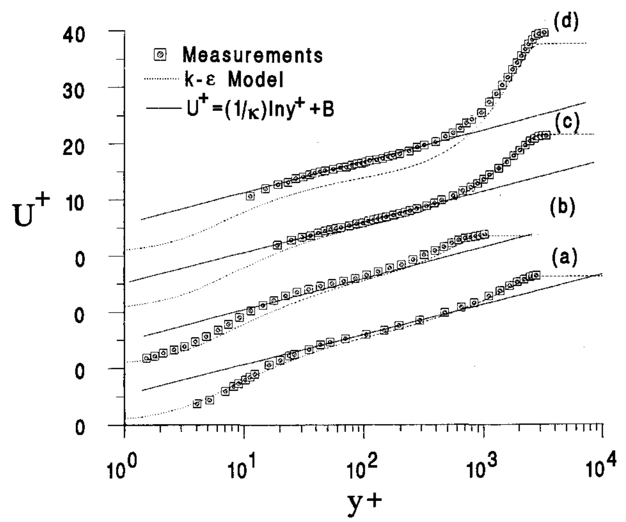

Fig. 1 Semilog plots of incompressible boundary layers in law of the wall coordinates; cases are as indicated in Table 1.

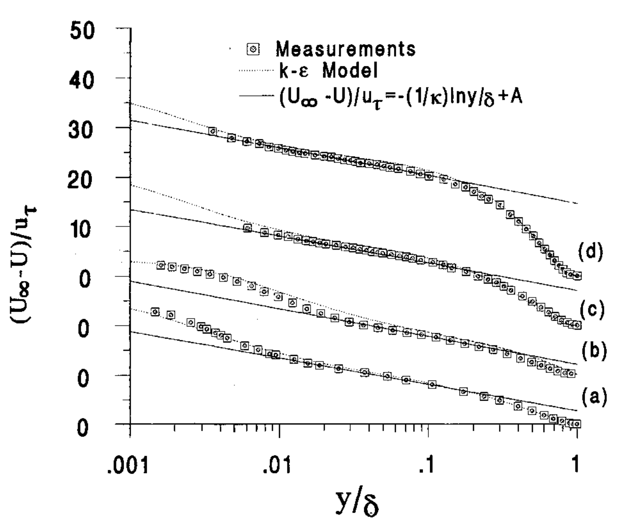

Fig. 2 Semilog plots of incompressible boundary layers in defect law coordinates; cases are as indicated in Table 1.

the value of A for the case of zero-pressure-gradient flow can be estimated once Δ and δ are known. According to Mellor and Gibson[16], Δ/δ is approximately 3.6 at a Reynolds number based on the displacement thickness, $Re = 5 \times 10^4$. Therefore, A is estimated to be about 2.52. Furthermore, Mellor and Gibson[16] have calculated the variations of A_m with β_c for equilibrium incompressible boundary layers. Since Eq. (9) is applicable to flows both with and without streamwise pressure gradients, and since β is related to β_c, once the variations of A_m with β_c are known, the variations of A with β can also be determined. Even though Δ is a well-defined length scale and δ is not, there is no loss of generality in using δ for the analysis just presented. In the results to follow, a consistent definition of δ is used to analyze the measurements and calculations. The measured δ is used to analyze measurements whereas the calculated δ is used to reduce the calculated mean velocity. In each case, δ is defined as the y location where $U = 0.99U_\infty$.

The validity and extent of applicability of Morkovin's hypothesis should be discussed before embarking on a verification of Eqs. (4) and (5). Simply stated, Morkovin's hypothesis amounts to the neglect of turbulent density fluctuations in supersonic boundary-layer analyses when the variables are decomposed using Reynolds averages. Consequently, when this hypothesis is valid, the crucial turbulence statistics are only altered by compressibility effects through changes in the mean density. Morkovin[9] analyzed the compressible boundary-layer measurements of Morkovin and Phinney[18] and Kistler[19] and arrived at two important conclusions: 1) the pressure fluctuations were quite small compared to density fluctuations and 2) the total temperature fluctuations were much smaller than the static temperature fluctuations in supersonic boundary layers with small rates of wall heat transfer. As a result, the Reynolds-averaged mean flow equations for supersonic boundary-layer flows could be reduced to a form similar to those derived for incompressible flows. This, in turn, allows the direct extension of incompressible turbulence closures to model supersonic turbulent flows. Under these conditions, it is generally believed that, for supersonic boundary layers with adiabatic wall conditions, the Morkovin hypothesis is valid for external Mach numbers M_∞ as large as 5. The calculations to be presented in the next section, however, indicate that its range of applicability may extend significantly beyond $M_\infty = 5$.

Results and Discussion

Boundary-layer measurements[4–6,11–13] are used to verify the logarithmic behavior deduced in Eqs. (4) and (5) for the wall layer and the defect layer. This data includes zero-pressure-gradient compressible boundary layers,[4–6] incompressible boundary layers with zero pressure gradient,[11,12] and flows with adverse pressure gradients.[13] First, comparisons are made with incompressible flow measurements to establish the values of κ, A, and B and the dependence of A on β. Second, comparisons are carried out with compressible boundary-layer measurements to verify the existence of an overlap region and to determine the values of κ, A, and B as well as the dependence of A and B on M_τ and B_q. The calculations of the compressible boundary layers[4–6] using Favre-averaged governing equations have already been reported by Zhang et al.[2,3] Since the two-equation model predictions are essentially identical to the second-order closure calculations, the present comparisons are solely drawn from the two-equation model results. Furthermore, the two-equation model of Zhang et al.[2] is used to calculate the incompressible boundary layers,[11–13] and the results are analyzed together with the predictions of the compressible boundary layers. Once the values of κ, A, and B are determined from the experimental data, the same procedure is used to deduce the values of these parameters from the model calculations. In the process, the ability of the models to predict compressible boundary-layer flows—particularly the logarithmic behavior—can be assessed.

Before proceeding to analyze the boundary-layer measurements and model calculations, a consistent procedure has to be adopted for the determination of κ, B, and A. First, the mean velocity data are plotted in the wall coordinates, U^+ vs y_w^+. In reducing the measured data, the reported skin friction coefficient, $C_f = 2\tau_w/\rho_\infty U_\infty^2$, is used to calculate the friction velocity u_τ. On the other hand, the predicted u_τ is used to analyze the calculated mean velocity. Then, the data points that fall on a straight line in a semilog plot are identified. The point closest to the wall is identified as $(y_w^+)_{in}$, whereas the point farthest away from the wall is denoted as $(y_w^+)_{out}$. These locations are determined for all cases examined, and they are listed in Table 1 for reference. It should be pointed out that, once these locations are determined for each case, they are then used to reduce both the experimental and calculated data for the values of κ, B, and A. Although this procedure may not yield the precisely correct values for κ, B, and A, if it is carried out consistently, the results should be correct qualitatively. In other words, the relative behavior of these parameters can be identified. A least squares fit of the data points bounded by $(y_w^+)_{in}$ and $(y_w^+)_{out}$ yields the slope κ^{-1} and the intercept B of the straight line. Finally, the data are plotted in the defect form, and again the same data points as before are used to define the straight line. In drawing the straight line in the defect plot, the slope is chosen to be the same as that determined from the law-of-the-wall plot. Thus, the intercept A is uniquely determined. This procedure of determining the parameters is used in all subsequent data analysis. The values of C_f, κ, B, and A obtained from experimental measurements and model calculations are listed in Table 1 for comparison. In Table 1, the values of the freestream Mach number M_∞, the ratio of the wall temperature to adiabatic wall temperature T_w/T_{aw}, and the momentum thickness Reynolds number Re_θ (where θ is the momentum thickness of the boundary layer) for each experiment are also listed. Therefore, $T_w/T_{aw} = 1.0$ for both isothermal incompressible flows and compressible flows with an adiabatic wall boundary condition.

Plots of the zero pressure gradient boundary-layer data of Klebanoff[12] and Karlsson and Johansson[11] in the wall-layer and defect-layer form are shown in Figs. 1 and 2, respectively. The val-

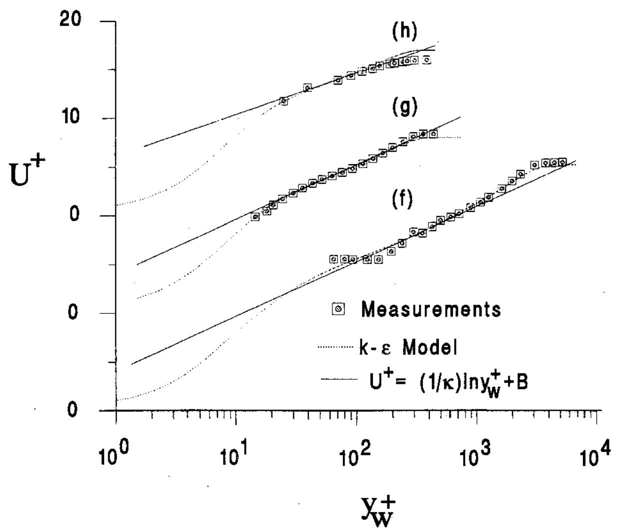

Fig. 3 Semilog plots of compressible boundary layers with an adiabatic wall in law of the wall coordinates; cases are as indicated in Table 1.

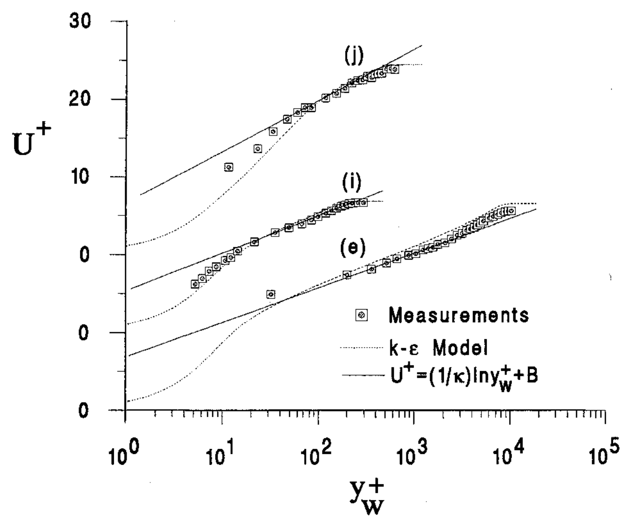

Fig. 5 Semilog plots of compressible boundary layers with constant wall temperature in law of the wall coordinates; cases are as indicated in Table 1.

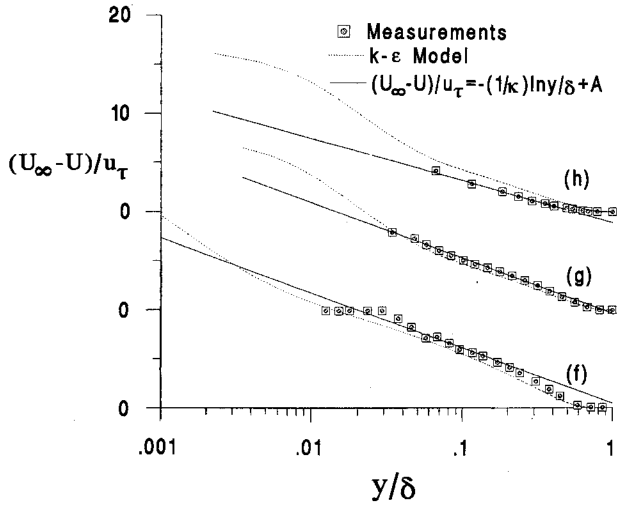

Fig. 4 Semilog plots of compressible boundary layers with an adiabatic wall in defect law coordinates; cases are as indicated in Table 1.

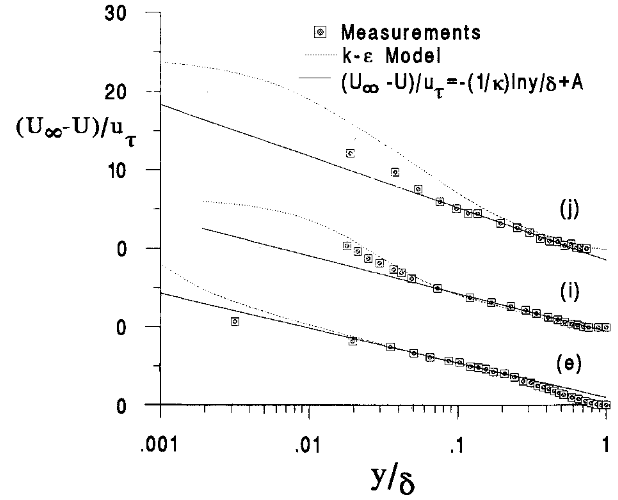

Fig. 6 Semilog plots of compressible boundary layers with constant wall temperature in defect law coordinates; cases are as indicated in Table 1.

ues of κ, B, and A, thus determined, are listed in Table 1. From Table 1, it can be seen that B and A vary with the Reynolds number; however, κ remains constant. This is expected from the analysis of Mellor.[20] According to Mellor and Gibson[16] and Eq. (9), the appropriate values of κ, B, and A at $Re = 5 \times 10^4$ for an isothermal, incompressible boundary-layer flow in equilibrium are 0.41, 4.9, and 2.52, respectively. The values of κ and A determined from Klebanoff's measurements[12] are in good agreement with $\kappa = 0.41$ and $A = 2.52$, whereas the two-equation model calculations yield results that are in fair agreement with the values determined from measurements. In addition, the two-equation model is capable of predicting the decrease of B and A with Reynolds numbers. This agreement is expected and simply serves to illustrate the consistency of the present strategy. Furthermore, this comparison shows that the choice of δ as the length scale in the defect layer is appropriate and does not contaminate the logarithmic behavior. The comparisons of the model calculations with the adverse pressure gradient data of Samuel and Joubert[13] are also shown in Figs. 1 and 2 with the law of the wall results given in Fig. 1 and the defect law results plotted in Fig. 2. Two stations, where one is near incipient separation, with different β are chosen to illustrate the dependence of A on β. Values of κ, B, and A determined from these plots are listed in Table 1. These results show that κ is not dependent on β, and its value thus determined is approximately 0.41; this offers further evidence for the logarithmic laws deduced in Eqs. (4) and (5). On the other hand, the dependence of A on β is clearly shown. Furthermore, the overlap between the wall layer and the defect layer does not disappear completely as separation approaches. Even at incipient separation, case d of Figs. 1 and 2, the overlap between the wall layer and the defect layer is still quite evident. The two-equation model is capable of reproducing these trends and, at the same time, gives a fairly correct prediction of the variation of A with β.

Six compressible flat plate boundary-layer cases are selected for analysis; they include one with wall heating (case e), three with an adiabatic wall (cases f–h), and two with wall cooling (cases i and j). The freestream Mach number varies from a low of 2.24 to a high of 10.31 (Table 1). Case e is taken from Spina and Smits,[6] case j is from Kussoy and Horstman,[5] whereas cases f–i are from the data compilation of Fernholz and Finley.[4] Case f from Fernholz and Finley is contributed by Shutts et al.,[21] case g is by Coles,[22] case h is by Watson et al.,[23] and case i is by Winkler and Cha.[24] For the sake of simplicity, these four cases are referenced to Fernholz and Finley[4] as the source in Table 1. All of these cases have been calculated using two-equation models and second-order closures, and the results are reported by Zhang et al.[2,3] It appears that there is good agreement with data when the mean velocity distributions are plotted either in the semilog form or in the linear form. According to Zhang et al.,[2,3] κ seems to remain fairly constant for compressible boundary-layer flows with an adiabatic wall boundary condition. However, a more critical analysis of the adiabatic wall boundary condition data sets along the present line reveals that κ is parametric in M_τ when M_∞ increases beyond 4.544.

The measurements and the calculated results are analyzed according to the procedure just outlined, and the values of κ, B, and A thus determined are listed in Table 1 for comparison. The logarithmic law-of-the-wall plots for the six cases examined are shown in Figs. 3 and 4, and the logarithmic defect-law plots are given in Figs. 5 and 6. Here, Figs. 3 and 5 show the plots for the compressible boundary layers with an adiabatic wall whereas Figs. 4 and 6 show those with constant wall temperature boundary conditions. It is obvious that an overlap exists for all zero-pressure-gradient compressible boundary layers studied. In other words, the rationale used to derive Eqs. (4) and (5) appears appropriate. From Table 1, it is clear that κ remains fairly constant for compressible boundary layers with an adiabatic wall where $M_\infty \leq 4.544$. This means that κ is not sensitive to M_τ in the external Mach number range, $0 \leq M_\infty \leq 4.544$. The reason could be due to the small value of M_τ, which only reaches 0.067 at $M_\infty = 2.24$. When $M_\infty = 10.31$, M_τ increases to 0.113. It is possible that as M_τ increases beyond 0.1, its influence on κ becomes more pronounced. The fact that a noticeable increase in κ is not detected in case g could be due to uncertainties in the measured data. Consequently, at high freestream Mach numbers, κ is found to be significantly different from its value determined at lower M_∞. As expected, the intercepts B and A are also parametric in M_τ; B increases with M_∞, whereas A decreases as M_∞ increases. The two-equation model does a fairly good job in the prediction of κ but a poor job of replicating the behavior of B and A. This is in spite of the rather accurate prediction of C_f.

In general, κ increases with M_∞ when the wall is heated but decreases when the wall is cooled. The variations are quite significant over the range of wall temperature ratios investigated. Also, κ is quite sensitive to variations in B_q. In the six cases examined, T_w / T_{aw} varies from a high of 1.11 to a low of 0.3. The value of κ thus deduced ranges from approximately 0.54 to about 0.35. Typically, κ is determined from the mean flow measurements that have an error margin of no more than 2–3%. For those experiments in which the wall shear stress was not measured directly, a Clauser plot technique[25] was used to determine u_τ, then κ is deduced using the procedure outlined earlier. This technique was found to be quite accurate for incompressible flow studies, and the u_τ thus determined agreed to within 5% of the direct measurements obtained by such techniques as force balance, wall shear probe, etc.[16,20,25,26] As a result, the uncertainty in the determination of κ would also be about 5%. This, however, is not the case for measurements in compressible boundary layers.[4–6] Measurement errors as large as 10% could occur in the measured mean flow velocity in supersonic flow experiments. Therefore, if κ is again determined using the given procedure, its error margin could be 10% or more. The present predictions are within or at the margin of these error limits. Their quantitative agreement with measurements is not good. Nevertheless, a clear trend is exhibited by the data and the model calculations replicate this behavior very well. On the other hand, a decrease of about 0.2 in κ over the range of T_w / T_{aw} investigated is estimated correctly by the two-equation model. The effect of total heat flux on B is clearly illustrated by comparing cases h and j where M_∞ differs by about 2 and T_w / T_{aw} decreases from 1.0 to 0.3. It can be seen that the value of B for case j is significantly lower than that for case h. The same trend is also observed for the behavior of A which decreases with decreasing T_w / T_{aw}. Again, the two-equation model results are in fair agreement with those determined from measurements. This means that the Morkovin hypothesis[9] is essentially valid for the bulk of the Mach number and wall temperature ratio range considered herein.

Since the values of κ, A, and B cannot be determined analytically,[10,15] there is no reason to expect them to be constant or to vary in any simple prescribed manner with the dimensionless variables of compressible flows for widely different M_∞ and T_w / T_{aw}. The value of κ for incompressible flows can be determined by comparison with measurements. Such an exercise has been carried out by numerous researchers. The generally accepted value for κ is 0.41 for incompressible flows both with and without streamwise pressure gradients.[17] This result is substantiated by the present analysis. On the other hand, the values of A and B are found to depend on the displacement thickness Reynolds number, even for incompressible flows.[16,18] For compressible boundary-layer flows over surfaces with adiabatic and isothermal wall boundary conditions, the dimensional similarity analysis presented shows that κ is parametric in γ, M_τ, and B_q, whereas the law of the wall intercept is parametric in Pr_w, γ, M_τ, and B_q; the defect-law intercept is dependent on β, γ, M_τ, and B_q. Again, these values cannot be determined analytically. Therefore, there is no compelling reason to assume that any one of these values should be constant for a fairly large range of M_∞ and T_w/T_{aw}. The present analysis strongly indicates that $\kappa \approx 0.41$ only for flat plate compressible boundary layers with an adiabatic wall where M_∞ is below 5. For other Mach numbers and wall thermal boundary conditions, κ is found to deviate significantly from its accepted incompressible value of 0.41. The intercepts, A and B, are also found to depend significantly on M_∞ and T_w/T_{aw}. Huang et al.[7] and Huang and Coleman[8] have demonstrated that the van Driest I transformation[1] can be used to transform the mean velocity yielding a compressible law of the wall with $\kappa \approx 0.41$ and $B = 5.2$. This form of the van Driest compressible law of the wall was found to be applicable to flat plate boundary-layer flows with adiabatic and cooled/heated walls for Mach numbers as high as 11. In Refs. 7 and 8, however, their analysis did not present evidence to show that an overlap exists between the wall layer and the defect layer using the transformed velocity. Despite this omission, their success is very encouraging, because it implies that the traditional van Driest transformation can be used to extend the incompressible law of the wall to its corresponding compressible form without having to alter the constants in the equation. However, it should not be interpreted as universally valid for all M_∞ and T_w/T_{aw}. Based on dimensional arguments and the overlap analysis presented herein, there is no reason to believe that κ can be parameterized by the values of $T_w/T_{\log}$ alone for cases with strong wall cooling or heating.

Skin Friction Law

A by-product of the preceding analysis is the skin friction law, which can be obtained by combining Eqs. (4) and (5). After much algebra, the result can be simplified to

$$\left(\frac{2}{C_f}\cdot\frac{T_\infty}{T_w}\right)^{1/2}-\frac{1}{2\kappa}\ln\frac{C_f}{2}=\frac{1}{\kappa}\ln Re_\theta$$

$$+\frac{1}{\kappa}\ln\left(\frac{T_\infty}{T_w}\right)^{n+1/2}+\frac{1}{\kappa}\ln\frac{\delta}{\theta}+B+A \tag{10}$$

Equation (10) shows that C_f is not only a function of the Reynolds number but is also affected by the wall temperature ratio T_∞/T_w and M_τ, B_q, β_c or β, γ, and Pr_w through the intercepts A and B. If A and B are known, Eq. (10) can be used to determine C_f. On the other hand, if Eqs. (4) and (5) are used to determine A and B as the present analysis has done, then Eq. (10) cannot be used to determine C_f because it is no longer an independent equation.

This skin friction law is different from the one deduced by Huang et al.[7] Their deduction invokes the assumptions of a mixing length, a van Driest-type damping function in the near-wall region and an empirical wake function obtained from incompressible boundary-layer data. As a result, the skin friction law obtained by Huang et al.[7] does not have an explicit $\ln(T_\infty/T_w)$ term in the equation. However, the effect of T_∞/T_w was accounted for in their algorithm used to deduce the skin friction. A similar expression for C_f has also been deduced by Rotta.[14] Rotta's skin friction law, however, contains an explicit $\ln(T_\infty/T_w)$ term. Furthermore, according to the analysis of Huang et al.,[7] C_f is relatively insensitive to the choice of Reynolds number, be it Re_θ or $Re_\theta(\mu_\infty/\mu_w)$. This observation is not quite consistent with the C_f expression given in Eq. (10). A few examples can be cited to illustrate this point. From the compressible boundary-layer measurements[4,5] listed in Table 1, the temperature ratio T_∞/T_w for cases (e–j) can be estimated to be 0.357, 0.526, 0.213, 0.031, 0.182, and 0.270, respectively. Therefore, $(n+1/2)\ln(T_\infty/T_w)$ varies from about -2 to about -7 depending on the value of n assumed. On the other hand, Re_θ varies from about 4000 to approximately 84,000 yielding a range for $\ln Re_\theta$ of approximately 8.3–11.3. These two terms are of comparable magnitude, and they are equally important in Eq. (10). In other words, the skin friction coefficient is also strongly dependent on the wall temperature ratio. The only assumptions invoked in the present analysis are the existence of a constant stress layer, the presence of an overlap between the wall layer and the defect layer, and the validity of the Markovin hypothesis. Therefore, the skin friction law given by Eq. (10) appears to be more physical.

Conclusions

Dimensional similarity arguments and the Morkovin hypothesis have been used to derive a general law of the wall and a general defect law for flat plate compressible boundary layers with adiabatic as well as heated and cooled wall thermal boundary conditions. When the assumption of an overlap is invoked, logarithmic behavior can be deduced for these two laws. The slopes of the logarithmic laws for the wall layer and the defect layer are equal and are given by κ^{-1}. This coefficient is not a universal constant, but rather, it depends parametrically on γ, M_τ, and B_q. The intercept B is found to depend on Pr_w, γ, M_τ, and B_q, whereas the intercept A depends parametrically on β, γ, M_τ, and B_q. Comparisons with measurements establish that κ assumes a constant value of approximately 0.41 for flat plate compressible boundary layers with M_∞ under 5 and adiabatic wall boundary conditions. Its value decreases as the wall temperature ratio decreases and increases with increasing wall temperature ratio and M_∞. An analysis was presented to show that the traditionally accepted van Driest compressible law of the wall renders $\kappa \propto (T_\infty/T_{\log})^{1/2}$, a result that is qualitatively consistent with the trends deduced in the present analysis. Although the results of Huang and Coleman[8] indicate that the van Driest law performs well for a fairly wide range of conditions, the overlap analysis presented in this study does raise questions about its universality (i.e., whether κ can be parameterized by $T_w/T_{\log}$ alone for all thermal wall conditions).

The intercepts A and B have also been found to be noticeably dependent on both the freestream Mach number and the wall temperature ratio. Both intercepts decrease with increasing M_∞ and decreasing T_w/T_{aw}. A by-product of this analysis is a skin friction law for compressible boundary layers. The skin friction coefficient thus deduced is found to depend on the flow Reynolds number, the wall temperature ratio, the pressure gradient parameter, and all other dimensionless parameters mentioned earlier.

Acknowledgments

Funding support from NASA Langley Research Center under Grant NAG-1-1080 is gratefully acknowledged.

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